Kuwait University | Math 101 | July 17, 2006
Math. & Comp. Sci. Dept. | Second Exam | Time: 90 min.

Calculators and Mobile Phones are not allowed.

1. Use differentials to approximate: $\sin(29°)$. **(3 Points)**

2. Find the equation of the normal line to the curve: $\tan(xy) + \sqrt{x+y} = 1$ at $x = 0$.

(4 Points)

3. A plate in a shape of a disk is heated. If the area A of the plate (in cm^2) after time t (in hours) is given by

$$A = \sqrt{t^2 + 3t + 6},$$

find the rate at which the radius of the plate is changing after two hours.

(4 Points)

4. a) State the Mean Value Theorem. **(1 Point)**

   b) Let f be a function such that $f'(x) < k, \ \forall x$, with $f(0) = 3$ and $f(3) = 6$. Find all possible values of k for such function to exist.

(3 Points)

5. Let $f(x) = \dfrac{3-x}{(x-1)^2}$, and given that $f'(x) = \dfrac{x-5}{(x-1)^3}$ and $f''(x) = \dfrac{2(7-x)}{(x-1)^4}$.

   a) Find the vertical and horizontal asymptotes for the graph of f, if any.

   b) Find the intervals on which the graph of f is increasing and the intervals on which the graph of f is decreasing. Find the local extrema of f, if any.

   c) Find the intervals on which the graph of f is concave upward and the intervals on which the graph of f is concave downward. Find the points of inflection, if any.

   d) Sketch the graph of f.

   e) Find the maximum and minimum values of f on [2,4].

(10 Points)

Answer KEY

- $\sin(29) = \sin(30+(-1°)) \approx \sin(30) + \cos(30)\,(-1)\frac{\pi}{180} = \frac{1}{2} + \frac{\sqrt{3}}{2}\left(-\frac{\pi}{180}\right) = \frac{1}{2} - \frac{\sqrt{3}\pi}{360}$

- when $x=0, y=1$. $\sec^2(xy)(y+xy') + \frac{1}{2}(x+y)^{-1/2}\cdot(1+y') = 0$

  @ $(0,1) \Rightarrow$ $\quad 1(1+0) + \frac{1}{2}\cdot(1+y') = 0 \Rightarrow y' = -3 \Rightarrow m_{nor} = \frac{1}{3}$

  $\Rightarrow y - 1 = \frac{1}{3}(x-0)$ or $y = \frac{1}{3}x + 1$

i- $A = \pi r^2 \Rightarrow \frac{dA}{dt} = 2\pi r\frac{dr}{dt} = \frac{1}{2}(t^2+3t+6)^{-1/2}\cdot(2t+3) \Rightarrow$

$\frac{dr}{dt} = \frac{2t+3}{2\sqrt{t^2+3t+6}}\cdot\frac{1}{2\pi r}$. $\qquad$ When $t=2$, $A=4$ & $r = \frac{2}{\sqrt{\pi}}$

$\therefore \left.\frac{dr}{dt}\right|_{t=2} = \frac{(2(2)+3)\sqrt{\pi}}{4\pi\sqrt{2^2+3(2)+6}\,(2)} = \frac{7}{32\sqrt{\pi}}$ cm/hr

4- a) MVT

b) Since $f'(x) < k \Rightarrow f$ is differentiable $\Rightarrow f$ is continuous. f is continuous on $[0,3]$ and differentiable on $(0,3)$, then by the MVT, $\exists\, c \in (0,3)$ such that $f'(c) = \frac{f(3)-f(0)}{3-0} = \frac{6-3}{3} = 1$. But since $f'(x) < k\ \forall x$ & since $f'(c) = 1$, then $k \in (1,\infty)$ for f to exists.

5- $D_f = \mathbb{R} \setminus \{1\}$

a) i) V.A: $\lim_{x\to1^-} f(x) = \lim_{x\to1^-}\frac{3-x}{(x-1)^2} = +\infty$; $\lim_{x\to1^+} f(x) = \lim_{x\to1^+}\frac{3-x}{(x-1)^2} = +\infty$

$\Rightarrow x = 1$ is a V.A.

ii) H.A: $\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty}\frac{3-x}{(x-1)^2} = 0$; $\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty}\frac{3-x}{(x-1)^2} = 0 \Rightarrow y = 0$ is H.A

b)

x	$-\infty$		1		5		$+\infty$
$f'(x) = \frac{x-5}{(x-1)^3}$		+	‖	−	0	+	
f	0	↗	$+\infty$ ‖ $+\infty$	↘	$-\frac{1}{8}$	↗	0

i) f is increasing on $(-\infty,1) \cup [5,+\infty)$

ii) f is decreasing on $(1,5]$

iii) $f(5) = -\frac{1}{8}$ is a local minimum

c) $f''(x) = \frac{2(7-x)}{(x-1)^4} = 0 \Rightarrow x = 7$.

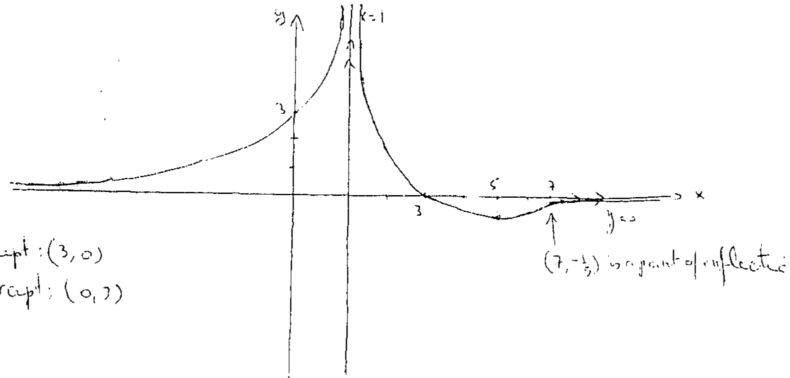
x	$-\infty$		1		7		$+\infty$
$f''(x)$		+	‖	+	0	−	
Concavity of f		∪	‖	∪		∩	

i) f is concave upward on $(-\infty,1) \cup (1,7]$

ii) f is concave downward on $[7,+\infty)$

iii) $(7,-\frac{1}{9})$ is the only point of inflection.

d)



x-intercept: $(3,0)$
y-intercept: $(0,3)$

e) $f'(x) = 0 \Rightarrow x = 5 \notin [2,4]$

$f(4) = -\frac{1}{9}$

$f(2) = 1$

Max. Value $= 1$

min. Value $= -1/9$